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CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

CONNECTED

WITH THE THEORY OF SURFACES

DISSERTATION

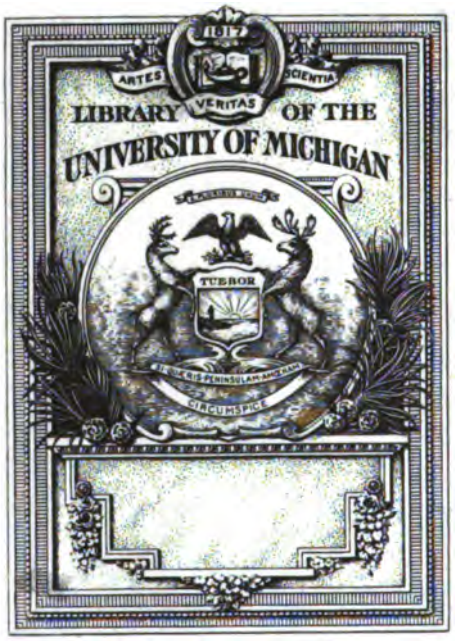
SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

NATHAN ALLEN PATTILLO

1897

The Lord Baltimore Press
THE FRIEDENWALD COMPANY
BALTIMORE, MD., U. S. A.



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INTRODUCTION.

In two notes in the Comptes Rendus of Oct. 26, 1896, and Nov. 16, 1896, and in the American Journal of Mathematics of Jan., 1897, Professor Craig has considered certain partial differential equations connected with the theory of surfaces.

In this paper the work done there has been continued. The first section is introductory, and gives a brief resumé of Laplace's method for the integration of linear partial differential equations of the second order.

In the next two sections Professor Craig's equations (E_{01}) and (E_{02}) are deduced together with their equivalent equations and the corresponding Laplace's series, and the general formulae for calculating their invariants are derived. The form of substitution required to pass from an equation of one series to one of the other series has also been found.

In the fourth section it is shown that by making the two equations (E_{01}) and (E_{02}) identical the lines of curvature form an isothermal system and the reciprocals of the two principal radii of curvature satisfy adjoint equations.

The next two sections are taken up with the calculation of the coefficients of the general transformed equations and making the general equations equal. Here the linear element reduces to the isothermal form.

In the seventh section the hypothesis $b_{01} = a_{02} = 0$, has been made. Here the equations (E_{01}) and (E_{02}) reduce to integrable forms and the lines of curvature form an isothermal system.

The following section has been devoted to the consideration of the hypothesis:

$$a_{01} = b_{02},$$
$$a_{02} = b_{01}.$$

Then the conditions have been found that must exist in order that the reciprocals of the principal radii of curvature of parallel surfaces may satisfy the same partial differential equations as those of the original surface.

The forms to which the equations (E_{01}) and (E_{02}) reduce for the ellipsoid and the cyclide of Dupin are then given and it is seen that they may be readily integrated by the method considered by Poisson.

I desire to make this acknowledgment of gratitude to Professor Craig, who suggested to me the subject of this dissertation and to whom I am indebted for valuable assistance.

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CERTAIN PARTIAL DIFFERENTIAL EQUATIONS CONNECTED WITH THE THEORY OF SURFACES.

1. Consider the linear partial differential equation

$$\frac{\partial^2\varphi}{\partial u\partial v}+a\frac{\partial\varphi}{\partial u}+b\frac{\partial\varphi}{\partial v}+c\varphi=0, \tag{1}$$

where a, b, c are any functions whatever of u and v; it can be put in either of the two following forms:

$$\left.\begin{aligned}\frac{\partial}{\partial u}\left(\frac{\partial\varphi}{\partial v}+a\varphi\right)+b\frac{\partial\varphi}{\partial v}+\left(c-\frac{\partial a}{\partial u}\right)\varphi=0\\ \frac{\partial}{\partial v}\left(\frac{\partial\varphi}{\partial u}+b\varphi\right)+a\frac{\partial\varphi}{\partial u}+\left(c-\frac{\partial b}{\partial v}\right)\varphi=0.\end{aligned}\right\} \tag{2}$$

Introduce the two functions h and k defined as follows:

$$\left.\begin{aligned}h=\frac{\partial a}{\partial u}+ab-c,\\ k=\frac{\partial b}{\partial v}+ab-c.\end{aligned}\right\} \tag{3}$$

If h is zero, the first of equations (2) can be written

$$\frac{\partial}{\partial u}\left(\frac{\partial\varphi}{\partial v}+a\varphi\right)+b\left(\frac{\partial\varphi}{\partial v}+a\varphi\right)=0.$$

Therefore, the complete determination of all the integrals of equation (1) is reduced to the successive integration of two equations of the first order

$$\left.\begin{aligned}\frac{\partial\varphi_1}{\partial u}+b\varphi_1=0,\\ \frac{\partial\varphi}{\partial v}+a\varphi=\varphi_1,\end{aligned}\right\} \tag{4}$$

integration which requires only quadratures.

In like manner, if k is zero, the second of equations (2) can be put in the form

$$\frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial u}+b\varphi\right)+a\left(\frac{\partial \varphi}{\partial u}+b\varphi\right)=0,$$

and we can replace equation (1) by the two equations

$$\left.\begin{aligned}\frac{\partial \varphi_{-1}}{\partial v}+a\varphi_{-1}&=0,\\ \frac{\partial \varphi}{\partial u}+b\varphi&=\varphi_{-1}.\end{aligned}\right\} \tag{5}$$

Suppose now that h and k are not zero. We shall show that these functions enjoy properties of invariance, which have a great importance in the study of the proposed equation. Consider successively the substitutions

$$\begin{aligned}&\varphi=\lambda\varphi';\\ &u=f(u'),\quad v=\psi(v');\\ &u=v',\qquad v=u'.\end{aligned}$$

Make at first

$$\varphi=\lambda\varphi', \tag{6}$$

λ being any function of u and v. The equation in φ will become

$$\left.\begin{aligned}\frac{\partial^2\varphi'}{\partial u\partial v}+\left(a+\frac{\partial \log\lambda}{\partial v}\right)\frac{\partial\varphi'}{\partial u}+\left(b+\frac{\partial\log\lambda}{\partial u}\right)\frac{\partial\varphi'}{\partial v}\\ +\left(c+a\frac{\partial\log\lambda}{\partial u}+b\frac{\partial\log\lambda}{\partial v}+\frac{1}{\lambda}\frac{\partial^2\lambda}{\partial u\partial v}\right)\varphi'=0.\end{aligned}\right\} \tag{7}$$

If we calculate the new values of h and k, we find that these functions have not changed.

Make now the substitution

$$u=f(u'),\quad v=\psi(v').$$

Equation (1) will become

$$\frac{\partial^2\varphi}{\partial u'\partial v'}+a\psi'(v')\frac{\partial\varphi}{\partial u'}+bf'(u')\frac{\partial\varphi}{\partial v'}+cf'(u')\psi'(v')\varphi=0.$$

The new values h', k' of h and k will be

$$\left.\begin{aligned}h'&=hf'(u')\psi'(v'),\\ k'&=kf'(u')\psi'(v').\end{aligned}\right\} \tag{8}$$

Finally the substitution

$$u=v',\quad v=u'$$

will exchange the values of h and k. All these properties justify the name of invariants which we shall give to h and k.

Suppose that the invariants h and k are not zero. Make the substitution given by the equation

$$\varphi_1 = \frac{\partial \varphi}{\partial v} + a\varphi. \tag{9}$$

Equation (1) can now be written

$$\frac{\partial \varphi_1}{\partial u} + b\varphi_1 = h\varphi. \tag{10}$$

If we eliminate φ between these two equations we get

$$\frac{\partial^2 \varphi_1}{\partial u \partial v} + a_1 \frac{\partial \varphi_1}{\partial u} + b_1 \frac{\partial \varphi_1}{\partial v} + c_1 \varphi_1 = 0, \tag{11}$$

where

$$a_1 = a - \frac{\partial \log h}{\partial v}, \quad b_1 = b, \quad c_1 = c - \frac{\partial a}{\partial u} + \frac{\partial b}{\partial v} - b\frac{\partial \log h}{\partial v}. \tag{12}$$

The value of the invariants will be

$$\left.\begin{aligned} h_1 &= 2h - k - \frac{\partial^2 \log h}{\partial u \partial v}, \\ k_1 &= h. \end{aligned}\right\} \tag{13}$$

They are expressed only as functions of the invariants of the given equation.

Make also the substitution

$$\varphi_{-1} = \frac{\partial \varphi}{\partial u} + b\varphi. \tag{14}$$

We shall have

$$\frac{\partial \varphi_{-1}}{\partial v} + a\varphi_{-1} = k\varphi, \tag{15}$$

and we get the following equation in φ_{-1}

$$\frac{\partial^2 \varphi_{-1}}{\partial u \partial v} + a_{-1} \frac{\partial \varphi_{-1}}{\partial u} + b_{-1} \frac{\partial \varphi_{-1}}{\partial v} + c_{-1} \varphi_{-1} = 0. \tag{16}$$

In this the coefficients have the values

$$a_{-1} = a, \quad b_{-1} = b - \frac{\partial \log k}{\partial u}, \quad c_1 = c - \frac{\partial b}{\partial v} + \frac{\partial a}{\partial u} - a\frac{\partial \log k}{\partial u}. \tag{17}$$

The values of the invariants will be

$$\left.\begin{aligned} h_{-1} &= k, \\ k_{-1} &= 2k - h - \frac{\partial^2 \log k}{\partial u \partial v}. \end{aligned}\right\} \tag{18}$$

Thus we deduce from the proposed equation (E) two new equations (E_1) and (E_{-1}). We can apply the same method to these two equations, but we will not get two new equations for each of them. The first of the two substitutions applied to the equation (E_{-1}) will give us the proposed equation in which φ will be replaced by $\frac{\varphi}{k}$. The second substitution, applied to (E_1), will bring us back to the proposed equation in which φ will be replaced by $\frac{\varphi}{h}$.

If then we regard as equivalent two equations which reduce to each othe by changing φ into $\lambda\varphi$ and which have, therefore, the same invariants, we see that the substitutions of Laplace applied successively will give only one linear series of equations

$$\ldots, (E_{-2}), (E_{-1}), (E), (E_1), (E_2), \ldots.$$

with positive and negative indices.

The invariants of equations (E_i) are deduced from each other by the repeated application of equations (13) and (18). We find thus

$$\left.\begin{aligned} h_{i+1} &= 2h_i - k_i - \frac{\partial^2 \log h_i}{\partial u \partial v}, \\ k_{i+1} &= h_i. \end{aligned}\right\} \tag{19}$$

Solved with respect to h_i and k_i, these formulas give

$$\left.\begin{aligned} h_i &= k_{i+1}, \\ k_i &= 2k_{i+1} - h_{i+1} - \frac{\partial^2 \log k_{i+1}}{\partial u \partial v}. \end{aligned}\right\} \tag{20}$$

We can give to i all integer values, positive or negative. Instead of considering two series of quantities h_i and k_i we could introduce only the quantities h_i. We shall then have the series

$$\ldots, h_{-2}, h_{-1}, h, h_1, h_2, \ldots,$$

deducing one from the other by the formula

$$h_{i+1}+h_{i-1}=2h_i-\frac{\partial^2 \log h_i}{\partial u \partial v}. \tag{21}$$

The invariants of the equation (E_i) will be h_i and h_{i-1}. By the linear combinations of equations (21) we can obtain the relation

$$h_{i+1}=h_i+h-k-\frac{\partial^2 \log h h_1 \ldots . h_i}{\partial u \partial v}. \tag{22}$$

2. Let u and v denote the parameters of the lines of curvature of a surface, ρ_1 and ρ_2 the principal radii of curvature of the surface at the point (u, v), ρ_1 being the principal radius corresponding to the line $v=$ const. (the u-line) and ρ_2 corresponding to $u=$ const. (the v-line). Let R_1, R_2 denote the radii of geodesic curvature of the lines $u=$ const. and $v=$ const. respectively.

We have now

$$\left.\begin{aligned} \rho_1&=\frac{E}{L}, \quad \rho_2=\frac{G}{N}, \\ \frac{1}{R_2}&=\frac{-1}{2E\sqrt{G}}\frac{\partial E}{\partial v}, \\ \frac{1}{R_1}&=\frac{-1}{2G\sqrt{E}}\frac{\partial G}{\partial u}. \end{aligned}\right\} \tag{23}$$

We have also the following equations connecting all of the preceding quantities

$$\left.\begin{aligned} \frac{1}{\rho_1^2}\frac{\partial \rho_1}{\partial v}+\frac{\sqrt{G}}{R_2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)&=0, \\ \frac{1}{\rho_2^2}\frac{\partial \rho_2}{\partial u}-\frac{\sqrt{E}}{R_1}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)&=0. \end{aligned}\right\} \tag{24}$$

Liouville's formula for the measure of curvature can be written

$$\frac{\sqrt{EG}}{\rho_1\rho_2}=\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_1}+\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_2}. \tag{25}$$

Substituting the values of ρ_1 and ρ_2 in equations (23), they can be reduced to the form

$$\left.\begin{aligned} \frac{\partial}{\partial v}\frac{\sqrt{E}}{\rho_1}&=-\frac{\sqrt{E}}{R_2}\frac{\sqrt{G}}{\rho_2}, \\ \frac{\partial}{\partial u}\frac{\sqrt{G}}{\rho_2}&=-\frac{\sqrt{G}}{R_1}\frac{\sqrt{E}}{\rho_1}. \end{aligned}\right\} \tag{26}$$

Differentiating the first of equations (24) for u and the second for v and eliminating $\frac{1}{\rho_2}$ and $\frac{1}{\rho_1}$ respectively, we obtain the equations

$$\left.\begin{aligned} \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} - \frac{\sqrt{G}}{R_2}\frac{\partial}{\partial u}\frac{1}{\rho_1} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial}{\partial v}\frac{1}{\rho_1} = 0, \\ \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_2} - \left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)\frac{\partial}{\partial u}\frac{1}{\rho_2} - \frac{\sqrt{E}}{R_1}\frac{\partial}{\partial v}\frac{1}{\rho_2} = 0. \end{aligned}\right\} \tag{27}$$

That is, $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are particular integrals of the differential equations

$$\left.\begin{aligned} (E_{01}) = \frac{\partial^2\varphi_{01}}{\partial u \partial v} - \frac{\sqrt{G}}{R_2}\frac{\partial\varphi_{01}}{\partial u} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial\varphi_{01}}{\partial v} = 0, \\ (E_{02}) = \frac{\partial^2\varphi_{02}}{\partial u \partial v} - \left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)\frac{\partial\varphi_{02}}{\partial u} - \frac{\sqrt{E}}{R_1}\frac{\partial\varphi_{02}}{\partial v} = 0. \end{aligned}\right\} \tag{28}$$

The differential equation satisfied by the coordinates (x, y, z) of the point (u, v) of the surface may now be written

$$\frac{\partial^2\varphi_0}{\partial u \partial v} + \frac{\sqrt{G}}{R_2}\frac{\partial\varphi_0}{\partial u} + \frac{\sqrt{E}}{R_1}\frac{\partial\varphi_0}{\partial v} = 0. \tag{29}$$

From equations (26) we derive in the same way as above the following:

$$\left.\begin{aligned} \frac{\partial^2\varphi_{01}}{\partial u \partial v} - \frac{\partial}{\partial u}\log\frac{\sqrt{E}}{R_2}\frac{\partial\varphi_{01}}{\partial v} - \frac{\sqrt{EG}}{R_1 R_2}\varphi_{01} = 0, \\ \frac{\partial^2\varphi_{02}}{\partial u \partial v} - \frac{\partial}{\partial v}\log\frac{\sqrt{G}}{R_1}\frac{\partial\varphi_{02}}{\partial u} - \frac{\sqrt{EG}}{R_1 R_2}\varphi_{02} = 0. \end{aligned}\right\} \tag{30}$$

These equations have $\frac{\sqrt{E}}{\rho_1}$ and $\frac{\sqrt{G}}{\rho_2}$ respectively as particular integrals. They are *equivalent* to equations (28); that is, they have the same invariants.

3. We shall now consider Laplace's series of equations corresponding to (E_{01}) and (E_{02}),

$$\left.\begin{aligned} \ldots, (E_{-t1}), \ldots (E_{-11}), (E_{01}), (E_{11}), (E_{21}), \ldots (E_{t1}), \ldots \\ \ldots, (E_{-t2}), \ldots (E_{-12}), (E_{02}), (E_{12}), (E_{22}), \ldots (E_{t2}), \ldots \end{aligned}\right\} \tag{31}$$

Using equations (3), we find the invariants of (E_{01}) are

$$h_{01} = \frac{\sqrt{EG}}{R_1 R_2}, \quad k_{01} = -\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{E}}{R_2} + \frac{\sqrt{EG}}{R_1 R_2}; \tag{32}$$

those of (E_{02}) are

$$h_{02} = -\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} + \frac{\sqrt{EG}}{R_1 R_2}, \quad k_{02} = \frac{\sqrt{EG}}{R_1 R_2}. \tag{33}$$

The application of formula (22) will give the general invariants. Thus we find for all positive values of i

$$\left.\begin{aligned} h_{i1} &= h_{i-1,1} + \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_2} - \frac{\partial^2}{\partial u \partial v} \log h_{01} h_{11} \ldots . h_{i-1,1} \\ &= h_{i-1,1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} h_{11} h_{21} \ldots . h_{i-1,1}, \\ h_{i2} &= h_{i-1,2} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} h_{02} h_{12} \ldots . h_{i-1,2}. \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} k_{i1} &= h_{i-1,1}, \\ k_{i2} &= h_{i-1,2}. \end{aligned}\right\} \tag{35}$$

For negative values of i we have

$$\left.\begin{aligned} h_{-i1} &= h_{-(i-1),1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_2} h_{-11} h_{-21} \ldots . h_{-(i-1),1}, \\ h_{-i2} &= h_{-(i-1),2} + \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} - \frac{\partial^2}{\partial u \partial v} \log h_{02} h_{-12} \ldots . h_{-(i-1),2} \\ &= h_{-(i-1),2} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_2} h_{-12} h_{-22} \ldots . h_{-(i-1),2}. \end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} k_{-i1} &= h_{-(i+1),1}, \\ k_{-i2} &= h_{-(i+1),2}. \end{aligned}\right\} \tag{37}$$

From the first of equations (34) it may be seen that

$$h_{11} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1}.$$

Therefore,

$$\left.\begin{aligned} h_{11} &= h_{02}, \\ k_{11} &= h_{01} = k_{02}. \end{aligned}\right\} \tag{38}$$

Consider the general invariants. Let us suppose

$$h_{i-1,1} = h_{i-2,2}; \tag{39}$$

then it may readily be seen that

$$\left.\begin{aligned} h_{i1} &= h_{i-1,2}, \\ k_{i1} &= k_{i-1,2}. \end{aligned}\right\} \tag{40}$$

But we have found equation (39) to be true for $i=2$; then it holds for all values of i. Therefore, we have the same invariants in the two series of equations (31), each invariant in the first series being equal to the preceding invariant in the second series. Then the equations (E_{i1}) and $(E_{i-1,2})$ are equivalent. Hence, the first may be reduced to the second by replacing the function φ_{i1} by $\lambda\varphi_{i-1,2}$. We can determine λ by identifying the coefficients in the two equations. By repeated application of equations (12) we find the general coefficients

$$\left.\begin{aligned} a_{i1} &= a_{01} - \frac{\partial \log h_{11}h_{11}\ldots h_{i-1,1}}{\partial v}, \\ a_{i-1,2} = a_{02} - \frac{\partial \log h_{02}\ldots h_{i-2,2}}{\partial v} &= a_{02} - \frac{\partial \log h_{11}\ldots h_{i-1,1}}{\partial v}, \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} b_{i1} &= b_{01}, \\ b_{i-1,2} &= b_{02}. \end{aligned}\right\} \tag{42}$$

The formulas of identification which we may take from equation (7) will give

$$\left.\begin{aligned} a_{i-1,2} &= a_{i1} + \frac{\partial \log \lambda}{\partial v}, \\ b_{i-1,2} &= b_{i1} + \frac{\partial \log \lambda}{\partial u}. \end{aligned}\right\} \tag{43}$$

Replacing $a_{i-1,2}$ and a_{i1} in the first equation by their values taken from (41), we get the equation

$$\frac{\partial \log \lambda}{\partial v} = \frac{\partial}{\partial v} \log \frac{\sqrt{G}}{R_2}.$$

After integration it may be placed in the form

$$\log \lambda = \log \frac{\sqrt{G}}{R_2} + \log U. \tag{44}$$

By substituting the values of $b_{i-1,2}$ and b_{i1} taken from (42) the second of equations (43) may be reduced to the form

$$\frac{\partial \log \lambda}{\partial u} = \frac{\partial}{\partial u} \log \frac{\sqrt{G}}{R_2}.$$

Hence,

$$\log \lambda = \log \frac{\sqrt{G}}{R_2} + \log V. \tag{45}$$

By comparing (44) and (45) it may be seen that U and V must be constants. Therefore, neglecting a constant factor, which would disappear after substitution, we may take

$$\lambda = \frac{\sqrt{G}}{R_2}.$$

Hence, if in (E_{i1}) the function φ_{i1} be replaced by $\frac{\sqrt{G}}{R_2}\varphi_{i-1,2}$, we shall find $(E_{i-1,2})$.

In the same way it may be shown that replacing φ_{-i2} in (E_{-i2}) by $\frac{\sqrt{E}}{R_1}\varphi_{-(i-1),1}$ will give the equation $(E_{-(i-1),1})$.

4. We shall now consider the result of making the equations (E_{01}) and (E_{02}) identical. By examining the coefficients it is evident that we have the condition

$$\left.\begin{aligned} \frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} &= 0, \\ \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} &= 0. \end{aligned}\right\} \tag{46}$$

The integral of the first of these equations may be written

$$\frac{\sqrt{G}}{R_2} = V.$$

Substituting the value of $\frac{\sqrt{G}}{R_2}$ from the first of (23), we have

$$-\frac{\partial}{\partial v}\log\sqrt{E} = V,$$

hence,

$$E = U_1 V_2. \tag{47}$$

From the second of equations (46) we get

$$\frac{\sqrt{E}}{R_1} = U.$$

This, combined with (23), gives

$$-\frac{\partial}{\partial u}\log\sqrt{G} = U;$$

we have then

$$G = U^2 V_1. \tag{48}$$

The linear element of the surface may now be written

$$ds^2 = U_2 V_2\left(\frac{U_1}{U_2}du^2 + \frac{V_1}{V_2}dv^2\right),$$

or

$$ds^2 = \lambda (U du^2 + V dv^2), \tag{49}$$

the U's being functions of u alone and the V's of v alone. Then the lines of curvature form an isothermal system.

The equations (28) both have now the form

$$\frac{\partial^2 \varphi}{\partial u \partial v} - \frac{\sqrt{G}}{R_2}\frac{\partial \varphi}{\partial u} - \frac{\sqrt{E}}{R_1}\frac{\partial \varphi}{\partial v} = 0, \tag{50}$$

which admits $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ as particular integrals. It may also be written

$$\frac{\partial^2 \varphi}{\partial u \partial v} + \frac{\partial \log \sqrt{E}}{\partial v}\frac{\partial \varphi}{\partial u} + \frac{\partial \log \sqrt{G}}{\partial u}\frac{\partial \varphi}{\partial v} = 0.$$

Since $E = \lambda U$ and $G = \lambda V$, we have

$$\frac{\partial^2 \varphi}{\partial u \partial v} - \frac{\partial \log \sqrt{\lambda}}{\partial v}\frac{\partial \varphi}{\partial u} - \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \varphi}{\partial v} = 0, \tag{51}$$

The equation satisfied by (x, y, z) of any point (u, v) of the surface is now

$$\frac{\partial^2 \varphi_0}{\partial u \partial v} + \frac{\partial \log \sqrt{\lambda}}{\partial v}\frac{\partial \varphi_0}{\partial u} + \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \varphi_0}{\partial v} = 0. \tag{52}$$

If we have a linear equation

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} + c\theta = 0,$$

its adjoint will be of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} - a\frac{\partial \theta}{\partial u} - b\frac{\partial \theta}{\partial v} + \left(c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v}\right)\theta = 0.$$

In equation (51) we have

$$a = -\frac{\partial \log \sqrt{\lambda}}{\partial v}, \quad b = -\frac{\partial \log \sqrt{\lambda}}{\partial u}, \quad c = 0,$$

hence,

$$c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} = 2\frac{\partial^2 \log \sqrt{\lambda}}{\partial u \partial v} = \frac{\partial^2 \log \lambda}{\partial u \partial v} = 0,$$

since $\lambda = U_1 V_1$. Then it follows that (52) is the adjoint of (51).

Therefore, if u and v are the parameters of lines of curvature of a surface and the linear element can be reduced to the isothermal form

$$ds^2 = \lambda(Udu^2 + Vdv^2), \quad (\lambda = U_2 V_2)$$

then the Cartesian coordinates (x, y, z) satisfy a differential equation which is the adjoint of that satisfied by the reciprocals of the principal radii of curvature.

The invariants of (51) are

$$h = k = \frac{1}{4}\frac{\partial \log \lambda}{\partial u}\frac{\partial \log \lambda}{\partial v} = \frac{1}{4}\frac{U_2'}{U_2}\cdot\frac{V_2'}{V_2}. \tag{53}$$

Therefore,

$$\frac{\partial^2 \log h}{\partial u \partial v} = 0.$$

We can find

$$(E_{11}) = \frac{\partial^2 \varphi_1}{\partial u \partial v} - \frac{\partial \log \sqrt{\lambda h}}{\partial v}\frac{\partial \varphi_1}{\partial u} - \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \varphi_1}{\partial v} + \frac{\partial \log \sqrt{\lambda}}{\partial u}\frac{\partial \log h}{\partial v}\varphi_1 = 0. \tag{54}$$

The invariants of this equation are

$$\left.\begin{aligned} h_1 &= \frac{1}{4}\frac{\partial \log \lambda}{\partial u}\frac{\partial \log \lambda}{\partial v} = h, \\ k_1 &= h. \end{aligned}\right\} \tag{55}$$

Then we have

$$h_1 = k_1 = h = k. \tag{56}$$

Then all the invariants in the series will be equal. Therefore, we are led to the consideration of only one equation, (E_{01}).

5. Make now the equations (E_{11}) and (E_{12}) identical. From equations (12) we obtain

$$a_{11} = -\frac{\sqrt{G}}{R_2} - \frac{\partial \log h_{01}}{\partial v},$$

$$a_{12} = -\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} - \frac{\sqrt{G}}{R_2} - \frac{\partial \log h_{02}}{\partial v}.$$

Equating these coefficients, we have

$$\frac{\partial}{\partial v}\log\frac{\sqrt{G}}{R_2} = \frac{\partial \log h_{02}}{\partial v}. \tag{57}$$

Making $b_{11} = b_{12}$ gives us

$$\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} = 0. \tag{58}$$

We have also

$$c_{11}=\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_2}-\frac{\partial^2}{\partial u\partial v}\log\frac{\sqrt{G}}{R_2}-\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}+\left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2}+\frac{\sqrt{E}}{R_1}\right)\frac{\partial\log h_{01}}{\partial v},$$
$$c_{12}=\frac{\partial^2}{\partial u\partial v}\log\frac{\sqrt{E}}{R_1}+\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_2}-\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}+\frac{\sqrt{E}}{R_1}\frac{\partial\log h_{02}}{\partial v}.$$

By equating these values and taking (58) into account we get

$$\frac{\partial^2\log h_{01}}{\partial u\partial v}=\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}. \tag{59}$$

But from (23) we see that

$$\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}=-\frac{\partial^2}{\partial u\partial v}\log\sqrt{G}.$$

Hence,

$$\frac{\partial^2}{\partial u\partial v}\log\sqrt{G}h_{01}=0. \tag{60}$$

The integral of (58) may be written

$$\frac{\sqrt{G}}{R_2}=V',$$

and this combined with (23) gives

$$-\frac{\partial\log\sqrt{E}}{\partial v}=V'.$$

Therefore, by integrating this, we have

$$E=U_1e^{-2\int V'}. \tag{61}$$

Since

$$\log\frac{\sqrt{G}}{R_2}=\log V',$$

the integration of equation (57) will give

$$\log V'+U=\log h_{02};$$

consequently

$$h_{02}=UV'. \tag{62}$$

We may write

$$h_{01}=\frac{\sqrt{EG}}{R_1R_2}=V'\frac{\sqrt{E}}{R_1}. \tag{63}$$

Equation (60) can now be put in the form

$$\frac{\partial^2}{\partial u \partial v} \log \sqrt{G} + \frac{\partial^2}{\partial u \partial v} \log V' + \frac{\partial^2}{\partial u \partial v} \log \sqrt{E} - \frac{\partial^2}{\partial u \partial v} \log R_1 = 0.$$

But from (58) we see that the second and third terms of this equation vanish. It then becomes

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} = 0.$$

It is evident then from (33) that

$$h_{12} = \frac{\sqrt{EG}}{R_1 R_2},$$

and consequently

$$h_{12} = h_{21}.$$

Combining this equation with (63), we get

$$h_{12} = V' \frac{\sqrt{E}}{R_1} = U' V'; \tag{64}$$

therefore,

$$\frac{\sqrt{E}}{R_1} = U'.$$

From (23) we see then that

$$-\frac{\partial}{\partial u} \log \sqrt{G} = U'.$$

The integration of this equation will give

$$G = V_1 e^{-2U'}. \tag{65}$$

Since $$h_{12} = h_{21} = U' V';$$

we obtain by differention

$$\frac{\partial^2 \log h_{21}}{\partial u \partial v} = 0.$$

But it has been found that

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} = 0;$$

and by substituting this in the preceding equation we find

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_2} = 0.$$

This value substituted in the second of equations (32) will reduce it to the form

$$k_{01} = \frac{\sqrt{EG}}{R_1 R_2}.$$

Clearly we have then

$$k_{01} = k_{02} = h_{01} = h_{02}, \tag{66}$$

that is, all the invariants in the two leading equations are now equal.

The linear element in this case will be expressed by the equation

$$ds^2 = e^{-2(U+V)}[U_1 e^{2U}\, du^2 + V_1 e^{2V} dv^2]. \tag{67}$$

If then u and v are the parameters of the lines of curvature of a surface and if the two first derived equations in the two series of Laplace (31) are identical, that is, $(E_{11}) = (E_{12})$; all the invariants in the two leading equations become equal to each other and the lines of curvature form an isothermal system.

6. We shall now consider the general case and make $(E_{i1}) = (E_{i2})$. To do this it is evident that we must have the coefficients respectively identical in the two equations.

We have as in (41)

$$a_{i1} = a_{01} - \frac{\partial}{\partial v} \log h_{01} h_{11} \ldots . h_{i-1,1},$$

$$a_{i2} = a_{02} - \frac{\partial}{\partial v} \log h_{02} h_{12} \ldots . h_{i-1,2},$$

$$= a_{02} - \frac{\partial}{\partial v} \log h_{11} h_{21} \ldots . h_{i1}.$$

Equating these values of the coefficients, we get the equation

$$a_{01} - \frac{\partial \log h_{01}}{\partial v} = a_{02} - \frac{\partial \log h_{i1}}{\partial v}. \tag{68}$$

But

$$a_{01} = -\frac{\sqrt{G}}{R_2},$$

and

$$a_{02} = -\left(\frac{\partial}{\partial v} \log \frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)$$

These values substituted in (68) reduce it to the form

$$\frac{\partial}{\partial v} \log h_{01} = \frac{\partial}{\partial v} \log \frac{\sqrt{E}}{R_1} + \frac{\partial}{\partial v} \log h_{i1};$$

therefore, since

$$h_{01} = \frac{\sqrt{EG}}{R_1 R_2},$$

we obtain the equation

$$\frac{\partial}{\partial v} \log \frac{\sqrt{G}}{R_2} = \frac{\partial}{\partial v} \log h_{t1}. \tag{69}$$

This is the first equation of condition for the identity proposed.

We know that

$$b_{t1} = b_{01},$$
$$b_{t2} = b_{02};$$

then making $b_{t1} = b_{t2}$,

gives us simply $b_{01} = b_{02}$.

Substituting the values of these coefficients, the equation becomes

$$\frac{\partial}{\partial u} \log \frac{\sqrt{G}}{R_2} = 0, \tag{70}$$

which is our second equation of condition.

By linear combinations of the last of equations (12) we may obtain the general coefficient

$$c_{t1} = c_{01} - \frac{\partial a_{01}}{\partial u} - \ldots - \frac{\partial a_{t-1,1}}{\partial u} + \frac{\partial b_{01}}{\partial v} + \ldots + \frac{\partial b_{t-1,1}}{\partial v} - b_{01} \frac{\partial \log h_{01} \ldots h_{t-1,1}}{\partial v} \tag{71}$$

Combinations of the first of equations (41) will give

$$-\frac{\partial a_{01}}{\partial u} - \frac{\partial a_{11}}{\partial u} - \ldots - \frac{\partial a_{t-1}}{\partial u} = -\frac{\partial^2}{\partial u \partial v} \log E^{\frac{t}{2}} + \frac{\partial^2}{\partial u \partial v} \log h_{01}^{t-1} h_{11}^{t-2} \ldots h_{t-2,1}, \tag{72}$$

since, from (23), we have

$$\frac{\sqrt{G}}{R_2} = -\frac{\partial}{\partial v} \log \sqrt{E}.$$

It may readily be seen also that

$$\frac{\partial b_{01}}{\partial v} + \frac{\partial b_{11}}{\partial v} + \ldots + \frac{\partial b_{t-1,1}}{\partial v} = -\frac{\partial^2}{\partial u \partial v} \log \frac{G^{\frac{t}{2}}}{R_2^t} + \frac{\partial^2}{\partial u \partial v} \log G^{\frac{t}{2}} = -\frac{\partial^2}{\partial u \partial v} \log \frac{1}{R_2^t}, \tag{73}$$

by remembering that

$$\frac{\sqrt{E}}{R_1} = -\frac{\partial}{\partial u}\log\sqrt{G}.$$

If these values in (72) and (73) be substituted now in equation (71), it may be put in the form

$$\left.\begin{aligned} c_{t1} &= c_{01} - \frac{\partial^2}{\partial u\partial v}\log\frac{E^{\frac{t}{2}}}{R_2^t} + \frac{\partial^2}{\partial u\partial v}\log h_{01}^{t-1}h_{11}^{t-2}\ldots.h_{t-2,1} \\ &\qquad - b_{01}\frac{\partial \log h_{01}\ldots.h_{t-1,1}}{\partial v} \\ &= c_{01} - \frac{\partial^2}{\partial u\partial v}\log\frac{\sqrt{E}}{R_2} + \frac{\partial^2}{\partial u\partial v}\log\frac{G^{\frac{t-1}{2}}}{R_1^{t-1}}h_{11}^{t-2}\ldots.h_{t-2,1} \\ &\qquad - b_{01}\frac{\partial \log h_{01}h_{11}\ldots.h_{t-1,1}}{\partial v}. \end{aligned}\right\} \quad (74)$$

But it may be seen from the first of equations (34) that proper linear combinations will give

$$h_{t-1,1} = \frac{\sqrt{EG}}{R_1R_2} - \frac{\partial^2}{\partial u\partial v}\log\frac{G^{\frac{t-2}{2}}}{R_1^t}h_{11}^{t-2}h_{21}^{t-2}\ldots.h_{t-2,1}, \quad (75)$$

or

$$\frac{\partial^2}{\partial u\partial v}\log\frac{G^{\frac{t-1}{2}}}{R_1^t}h_{11}^{t-2}h_{21}^{t-2}\ldots.h_{t-2,1} = \frac{\sqrt{EG}}{R_1R_2} - h_{1-1,1}.$$

If we substitute this value in equation (74), we have finally

$$c_{t1} = c_{01} + k_{01} - h_{t-1,1} - b_{01}\frac{\partial \log h_{01}h_{11}\ldots.h_{t-1,1}}{\partial v}. \quad (76)$$

In the same manner it may be shown that in the second series of equations (31) the general coefficient is

$$\left.\begin{aligned} c_{t2} &= c_{02} + k_{02} - h_{t-1,2} - b_{02}\frac{\partial \log h_{02}h_{12}\ldots.h_{t-1,2}}{\partial v} \\ &= c_{02} + h_{01} - h_t - b_{02}\frac{\partial \log h_{11}h_{21}\ldots.h_{t1}}{\partial v}, \end{aligned}\right\} \quad (77)$$

by remembering the relations which exist between the invariants of the two series.

Since

$$b_{01} = b_{02},$$
$$c_{01} = c_{02} = 0,$$

if we place

$$c_{t1} = c_{t2},$$

we will get after obvious reductions the following equation

$$k_{01} - h_{i-1,1} - b_{01}\frac{\partial \log h_{01}}{\partial v} = h_{01} - h_{i1} - b_{01}\frac{\partial \log h_{i1}}{\partial v}. \tag{78}$$

But we know

$$k_{01} - h_{01} = -\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{E}}{R_2},$$

and from (34)

$$h_{i1} - h_{i-1,1} = -\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{G}}{R_1}h_{11}h_{21}\ldots.h_{i-1,1}.$$

Putting these values in equation (78), it may now be written

$$-\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{E}}{R_2} - \frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{G}}{R_1}h_{11}h_{21}\ldots.h_{i-1,1}$$
$$-b_{01}\frac{\partial \log h_{01}}{\partial v} + b_{01}\frac{\partial \log h_{i1}}{\partial v} = 0. \tag{79}$$

On account of the second equation of condition (70) we have

$$b_{01} = -\frac{\sqrt{E}}{R_1},$$

and from the first condition for the proposed identity we found

$$\frac{\partial}{\partial v}\log h_{i1} = \frac{\partial}{\partial v}\log\frac{\sqrt{G}}{R_2}.$$

Introducing these values in equation (79) above and making slight reductions, it becomes

$$-\frac{\partial^2}{\partial u \partial v}\log h_{01}h_{11}\ldots.h_{i-1,1} + \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1} = 0.$$

Since

$$\frac{\sqrt{E}}{R_1} = -\frac{\partial}{\partial u}\log\sqrt{G},$$

the last equation may now be written in the form

$$\frac{\partial^2}{\partial u \partial v}\log\sqrt{G}\,h_{01}h_{11}\ldots.h_{i-1,1}, \tag{80}$$

which results from making the coefficients c_{i1} and c_{i2} identical.

The integration of the second equation of condition (70) will give

$$\log\frac{\sqrt{G}}{R_2} = \log V',$$

or

$$\frac{\sqrt{G}}{R_2} = V'.$$

But from (23) we have

$$\frac{\sqrt{G}}{R_2} = -\frac{\partial}{\partial v}\log\sqrt{E};$$

hence

$$-\frac{\partial}{\partial v}\log\sqrt{E} = V'.$$

By integrating the last equation we get

$$E = U_{1s}^{-2}V. \tag{81}$$

The equation

$$\frac{\partial}{\partial v}\log h_{i1} = \frac{\partial}{\partial v}\log\frac{\sqrt{G}}{R_2}$$

may be integrated and placed in the form

$$\log h_{i1} = \log\frac{\sqrt{G}}{R_2} + \log U' = \log V' + \log U',$$

therefore

$$h_{i1} = U'V'. \tag{82}$$

Equation (80) may be written as follows:

$$\frac{\partial^2}{\partial u\partial v}\log\sqrt{G} + \frac{\partial^2}{\partial u\partial v}\log\sqrt{E} - \frac{\partial^2}{\partial u\partial v}\log R_1 + \frac{\partial^2}{\partial u\partial v}\log\frac{\sqrt{G}}{R_2} + \frac{\partial^2}{\partial u\partial v}\log h_{11}h_{21}\ldots.h_{i-1,1} = 0.$$

But it has previously been shown that the second and fourth terms of this equation are equal to zero. It then becomes

$$\frac{\partial^2}{\partial u\partial v}\log\frac{\sqrt{G}}{R_1} + \frac{\partial^2}{\partial u\partial v}\log h_{11}h_{21}\ldots.h_{i-1,1} = 0. \tag{83}$$

From (34) we know that

$$h_{i1} = h_{i-1,1} - \frac{\partial^2}{\partial u\partial v}\log\frac{\sqrt{G}}{R_1} - \frac{\partial^2}{\partial u\partial v}\log h_{11}h_{21}\ldots.h_{i-1,1}.$$

Hence, by combining this with the equation above, we get

$$h_{i1} = h_{i-1,1}.$$

But we have found

$$h_{i1} = U'V';$$

consequently

$$h_{i-1,1} = U'V'. \tag{84}$$

The differentiation of the last equation will give

$$\frac{\partial^2}{\partial u \partial v} \log h_{i-1,1} = 0, \tag{85}$$

and, if this be substituted in (83), we have

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} + \frac{\partial^2}{\partial u \partial v} \log h_{11} h_{21} \ldots . h_{i-2,1} = 0. \tag{86}$$

Again

$$h_{i-1,1} = h_{i-2,1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} - \frac{\partial^2}{\partial u \partial v} \log h_{11} h_{21} \ldots . h_{i-2,1};$$

and, consequently

$$h_{i-1,1} = h_{i-2,1}.$$

Comparing this with equation (85), we have

$$\frac{\partial^2}{\partial u \partial v} \log h_{i-2,1} = 0.$$

Substituting, as before, (86) now becomes

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} + \frac{\partial^2}{\partial u \partial v} \log h_{11} h_{21} \ldots . h_{i-3,1} = 0. \tag{87}$$

Thus, it may easily be seen that by continuing this process, annuling one term each time, we are left finally with the equation

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} = 0.$$

Then we have

$$h_{11} = h_{01} = h_{02}.$$

If we should make the equations (E_{01}) and (E_{11}) identical, it would give

$$h_{01} = h_{11},$$
$$k_{01} = k_{11}.$$

But from the relations of the invariants in the two series we have

$$h_{11} = h_{02},$$
$$k_{11} = k_{02};$$

hence

$$h_{02} = h_{01} = h_{11},$$
$$k_{02} = k_{01} = k_{11}.$$

Then, by making the equations (E_{01}) and (E_{02}) identical, we find that all the invariants in the two series are equal and we are led only to the consideration of

$$(E_{01}) = (E_{11}).$$

Since

$$\left.\begin{aligned} h_{01} &= h_{02}, \\ k_{01} &= k_{02}, \end{aligned}\right\} \tag{88}$$

we have

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} = 0,$$
$$\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_2} = 0.$$

The integrals of these two equations may be written in the form

$$\frac{\sqrt{G}}{R_1} = U_2 V_1,$$
$$\frac{\sqrt{E}}{R_2} = U_1 V_2.$$

Substituting the values of $\frac{\sqrt{G}}{R_1}$ and $\frac{\sqrt{E}}{R_2}$ from equations (23), we get

$$-\frac{1}{2\sqrt{EG}} \frac{\partial G}{\partial u} = U_2 V_1,$$
$$-\frac{1}{2\sqrt{EG}} \frac{\partial E}{\partial v} = U_1 V_2.$$

The elimination of $-\frac{1}{2\sqrt{EG}}$ from these equations will give

$$U_2 V_1 \frac{\partial E}{\partial v} = U_1 V_2 \frac{\partial G}{\partial u},$$

or

$$\frac{U_2}{U_1} \frac{\partial E}{\partial v} = \frac{V_2}{V_1} \frac{\partial G}{\partial u},$$

where the $U'S$ are functions of u alone and the $V'S$ of v alone.

Write now

$$U_2 = UU_1,$$
$$V_2 = VV_1;$$

then the equation above becomes

$$\frac{\partial}{\partial v}(UE) = \frac{\partial}{\partial u}(VG). \tag{89}$$

But this is a necessary condition for the existence of an exact differential equation

$$(UE)\,du + (VG)\,dv = d\varphi, \tag{90}$$

where

$$E = \frac{1}{U}\frac{\partial \varphi}{\partial u},$$
$$G = \frac{1}{V}\frac{\partial \varphi}{\partial v}.$$

Take u and v arbitrary and φ will have to determined.

From (23) we have

$$\frac{\sqrt{G}}{R_2} = -\frac{\partial}{\partial v}\log\sqrt{E};$$

and, by introducing the value of E from the first of the two equations above, it becomes

$$\frac{\sqrt{G}}{R_2} = -\frac{\partial}{\partial v}\left(-\frac{1}{2}\log U + \frac{1}{2}\log\frac{\partial \varphi}{\partial u}\right)$$
$$= -\frac{1}{2}\frac{\partial}{\partial v}\log\frac{\partial \varphi}{\partial u} = -\frac{1}{2}\frac{\dfrac{\partial^2 \varphi}{\partial u \partial v}}{\dfrac{\partial \varphi}{\partial u}}.$$

In like manner we can find

$$\frac{\sqrt{E}}{R_1} = -\frac{1}{2}\frac{\dfrac{\partial^2 \varphi}{\partial u \partial v}}{\dfrac{\partial \varphi}{\partial v}}.$$

The products of these two equations will give

$$\frac{\sqrt{EG}}{R_1 R_2} = \frac{1}{4}\frac{\left(\dfrac{\partial^2 \varphi}{\partial u \partial v}\right)^2}{\dfrac{\partial \varphi}{\partial u}\dfrac{\partial \varphi}{\partial v}} = UU_1^2 VV_1^2. \tag{91}$$

If we place

$$W = e^{2\int UU_1^2 du + 2\int VV_1^2 dv},$$

we have then

$$\frac{\partial \log W}{\partial u} = 2\,UU_1^2,$$
$$\frac{\partial \log W}{\partial v} = 2VV_1^2.$$

Hence, equation (91) may now be placed in the form

$$\frac{\left(\frac{\partial^2 \varphi}{\partial u \partial v}\right)^2}{\frac{\partial \varphi}{\partial u}\frac{\partial \varphi}{\partial v}} = \frac{\partial \log W}{\partial u}\frac{\partial \log W}{\partial v}, \tag{92}$$

or

$$\frac{\partial}{\partial v}\log\frac{\partial \varphi}{\partial u}\cdot\frac{\partial}{\partial u}\log\frac{\partial \varphi}{\partial v} = \frac{\partial \log W}{\partial u}\cdot\frac{\partial \log W}{\partial v}.$$

7. We propose now to consider the result of making $b_{01} = a_{03} = 0$, that is

$$\left.\begin{aligned}\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1} = 0,\\ \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2} = 0.\end{aligned}\right\} \tag{93}$$

By substituting the values found in (23) the two equations above easily reduce to

$$\frac{\partial}{\partial u}\log R_2 = 0,$$
$$\frac{\partial}{\partial v}\log R_1 = 0.$$

The integrals of these equations are

$$\left.\begin{aligned}R_2 = f(v),\\ R_1 = \Psi(u).\end{aligned}\right\} \tag{94}$$

We may, therefore, write

$$-\frac{1}{R_1} = \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u} = U,$$
$$-\frac{1}{R_2} = \frac{1}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v} = V,$$

where U is a function only of u and V a function of v only. By eliminating $\frac{1}{\sqrt{EG}}$ we obtain the equation

$$V\frac{\partial\sqrt{G}}{\partial u}=U\frac{\partial\sqrt{E}}{\partial v},$$

or

$$\frac{\partial}{\partial u}(V\sqrt{G})=\frac{\partial}{\partial v}(U\sqrt{E}). \tag{95}$$

We see that this is a condition for the exact differential equation

$$(U\sqrt{E})\,du+(V\sqrt{G})\,dv=d\varphi, \tag{96}$$

where we have

$$\sqrt{E}=\frac{1}{U}\frac{\partial\varphi}{\partial u},$$

$$\sqrt{G}=\frac{1}{V}\frac{\partial\varphi}{\partial v}.$$

We find as before

$$\frac{\sqrt{G}}{R_2}=-\frac{\frac{\partial^2\varphi}{\partial u\partial v}}{\frac{\partial\varphi}{\partial u}},$$

$$\frac{\sqrt{E}}{R_1}=-\frac{\frac{\partial^2\varphi}{\partial u\partial v}}{\frac{\partial\varphi}{\partial v}},$$

and, therefore, the differential equation

$$\frac{\partial^2\varphi}{\partial u\partial v}=\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}. \tag{97}$$

The integral of this may be written

$$\varphi=-\log(U_1+V_1). \tag{98}$$

It may then easily be seen that

$$E=\frac{U_1'^2}{U^2(U_1+V_1)^2},$$

$$G=\frac{V_1'^2}{V^2(U_1+V_1)^2}.$$

The linear element may now be expressed by the equation

$$ds^2=\frac{1}{(U_1+V_1)^2}\left[\frac{U_1'^2}{U^2}\,du^2+\frac{V_1'^2}{V^2}\,dv^2\right] \tag{99}$$

or
$$ds^2 = \frac{1}{\lambda}(Udu^2 + Vdv^2).$$

This is a form which characterizes isothermal systems.

Then, if u and v are the parameters of lines of curvature of a surface, and if the radius of geodesic curvature of the line $u = \text{const.}$ is a function of u alone and that of $v = \text{const.}$ of v alone, the lines of curvature of the surface will form an isothermal system.

The leading equations of the two series now become

$$\left.\begin{aligned}(E_{01}) &= \frac{\partial^2\varphi_{01}}{\partial u\partial v} - \frac{\sqrt{G}}{R_2}\frac{\partial\varphi_{01}}{\partial u} = 0,\\ (E_{02}) &= \frac{\partial^2\varphi_{02}}{\partial u\partial v} - \frac{\sqrt{E}}{R_1}\frac{\partial\varphi_{02}}{\partial v} = 0,\end{aligned}\right\} \tag{100}$$

while that satisfied by (x, y, z) remains

$$(E) = \frac{\partial^2\varphi}{\partial u\partial v} + \frac{\sqrt{G}}{R_2}\frac{\partial\varphi}{\partial u} + \frac{\sqrt{E}}{R_1}\frac{\partial\varphi}{\partial v} = 0. \tag{101}$$

Since
$$\sqrt{\lambda} = U_1 + V_1,$$

we obtain
$$\begin{aligned}\frac{\sqrt{G}}{R_2} &= -\frac{\partial\log\sqrt{E}}{\partial v} = \frac{V_1'}{U_1 + V_1} = \frac{\partial\log\sqrt{\lambda}}{\partial v},\\ \frac{\sqrt{E}}{R_1} &= -\frac{\partial\log\sqrt{G}}{\partial u} = \frac{U_1'}{U_1 + V_1} = \frac{\partial\log\sqrt{\lambda}}{\partial u}.\end{aligned}$$

By substituting these in the equations above we get

$$\left.\begin{aligned}(E_{01}) &= \frac{\partial^2\varphi_{01}}{\partial u\partial v} - \frac{\partial\log\sqrt{\lambda}}{\partial v}\frac{\partial\varphi_{01}}{\partial u} = 0,\\ (E_{02}) &= \frac{\partial^2\varphi_{02}}{\partial u\partial v} - \frac{\partial\log\sqrt{\lambda}}{\partial u}\frac{\partial\varphi_{02}}{\partial v} = 0,\end{aligned}\right\} \tag{102}$$

$$(E) = \frac{\partial^2\varphi}{\partial u\partial v} + \frac{\partial\log\sqrt{\lambda}}{\partial v}\frac{\partial\varphi}{\partial u} + \frac{\partial\log\sqrt{\lambda}}{\partial u}\frac{\partial\varphi}{\partial v} = 0. \tag{103}$$

The invariants of the two proposed equations are now

$$\left.\begin{aligned}h_{01} &= -\frac{\partial^2}{\partial u\partial v}\log\sqrt{\lambda},\\ k_{01} &= 0,\end{aligned}\right\} \tag{104}$$

$$\left.\begin{aligned}h_{02} &= 0,\\ k_{02} &= -\frac{\partial^2}{\partial u\partial v}\log\sqrt{\lambda}.\end{aligned}\right\} \tag{105}$$

That is, they have their invariants equal but taken in opposite order.

8. Next we shall consider the result of making the coefficients of the equations (E_{01}) and (E_{02}) equal but taken in opposite order, that is

$$a_{01} = b_{02},$$
$$a_{02} = b_{01}.$$

By substituting their values we have

$$\left.\begin{aligned} \frac{\sqrt{G}}{R_2} &= \frac{\sqrt{E}}{R_1}, \\ \frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1} &= \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}. \end{aligned}\right\} \tag{106}$$

By combining these we get

$$\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} = \frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1},$$

or

$$\frac{R_2}{\sqrt{G}}\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_2} = \frac{R_1}{\sqrt{E}}\frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}.$$

Taking into account the first of equations (106), we have

$$\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_2} = \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_1}. \tag{107}$$

But

$$\frac{\sqrt{G}}{R_2} = -\frac{\partial}{\partial v}\log\sqrt{E},$$
$$\frac{\sqrt{E}}{R_1} = -\frac{\partial}{\partial u}\log\sqrt{G}.$$

These values placed in equation (107) will give

$$\frac{\partial^2}{\partial u \partial v}\log\sqrt{E} = \frac{\partial^2}{\partial u \partial v}\log\sqrt{G}.$$

If we integrate this equation, we shall get

$$E = G \,.\, U \,.\, V, \tag{108}$$

where U is a function of u only and V of v only. The linear element can now be put in the form

$$ds^2 = G(UVdu^2 + dv^2),$$

or

$$ds^2 = GV(Udu^2 + V_1 dv^2). \tag{109}$$

Then the lines of curvature form an isothermal system.

9. Let us now consider the forms to which the equations (E_{01}) and (E_{02}) reduce for surfaces parallel to the original one.

Since the surface is referred to its lines of curvature we have the following values for parallel surfaces:

$$\left.\begin{aligned} E_0 &= E\left(1-\frac{a}{\rho_1}\right)^2 = E\left(\frac{\rho_1^0}{\rho^1}\right)^2, \\ G_0 &= G\left(1-\frac{a}{\rho_2}\right)^2 = G\left(\frac{\rho_2^0}{\rho_2}\right)^2, \end{aligned}\right\} \tag{110}$$

$$\left.\begin{aligned} \rho_1^0 &= \rho_1 - a, \\ \rho_2^0 &= \rho_2 - a, \end{aligned}\right\} \tag{111}$$

where a is a constant which changes as we pass from one surface to the next one.

We have now

$$\left.\begin{aligned} \frac{\partial}{\partial u}\frac{1}{\rho_1} &= \left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2 \frac{\partial}{\partial u}\frac{1}{\rho_1^0}, \\ \frac{\partial}{\partial v}\frac{1}{\rho_1} &= \left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2 \frac{\partial}{\partial v}\frac{1}{\rho_1^0}. \end{aligned}\right\} \tag{112}$$

By differentiating the first of these equations for v and the second for u we get

$$\left.\begin{aligned} \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} &= \left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2 \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1^0} + \frac{\partial}{\partial v}\left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2 \frac{\partial}{\partial u}\frac{1}{\rho_1^0}, \\ \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} &= \left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2 \frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1^0} + \frac{\partial}{\partial u}\left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2 \frac{\partial}{\partial v}\frac{1}{\rho_1^0}. \end{aligned}\right\} \tag{113}$$

By substituting these values from (112) and (113) the equation

$$\frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} - \frac{\sqrt{G}}{R_2}\frac{\partial}{\partial u}\frac{1}{\rho_1} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial}{\partial v}\frac{1}{\rho_1} = 0,$$

can be put in either of the two following forms:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1^0} - \left[\frac{\sqrt{G}}{R_2} - \frac{\partial}{\partial v}\log\left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2\right]\frac{\partial}{\partial u}\frac{1}{\rho_1^0} \\ &\qquad - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial}{\partial v}\frac{1}{\rho_1^0} = 0, \\ &\frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1^0} - \frac{\sqrt{G}}{R_2}\frac{\partial}{\partial u}\frac{1}{\rho_1^0} - \left[\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right. \\ &\qquad \left. - \frac{\partial}{\partial u}\log\left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2\right]\frac{\partial}{\partial v}\frac{1}{\rho_1^0} = 0. \end{aligned}\right\} \tag{114}$$

By examining these we see that in order that these equations reduce to the same form as the original equation, we must have

$$\left.\begin{aligned}\frac{\partial}{\partial v}\log\left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2=0,\\ \frac{\partial}{\partial u}\log\left(\frac{\rho_1^0}{\rho_1^0+a}\right)^2=0.\end{aligned}\right\} \tag{115}$$

The integration of these will give

$$\left.\begin{aligned}\rho_1^0=\varphi(u),\\ \rho_1^0=\varphi_1(v).\end{aligned}\right\} \tag{116}$$

In the same manner as above the equation (E_{02}) may be put in either of the forms

$$\left.\begin{aligned}\frac{\partial^2}{\partial u\partial v}\frac{1}{\rho_2^0}-\left[\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1}+\frac{\sqrt{G}}{R_2}-\frac{\partial}{\partial v}\log\left(\frac{\rho_2^0}{\rho_2^0+a}\right)^2\right]\frac{\partial}{\partial u}\frac{1}{\rho_2^0}\\ -\frac{\sqrt{E}}{R_1}\frac{\partial}{\partial v}\frac{1}{\rho_2^0}=0,\\ \frac{\partial^2}{\partial u\partial v}\frac{1}{\rho_2^0}-\left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1}+\frac{\sqrt{G}}{R_2}\right)\frac{\partial}{\partial u}\frac{1}{\rho_2^0}\\ -\left[\frac{\sqrt{E}}{R_1}-\frac{\partial}{\partial u}\log\left(\frac{\rho_2^0}{\rho_2^0+a}\right)^2\right]\frac{\partial}{\partial v}\frac{1}{\rho_2^0}=0.\end{aligned}\right\} \tag{117}$$

The condition that these reduce to the same form as (E_{02}) is, as before,

$$\left.\begin{aligned}\rho_2^0=\varphi(u),\\ \rho_2^0=\varphi_1(v).\end{aligned}\right\} \tag{118}$$

Then if each of the principal radii of curvature of the parallel surface is a function of only one of the parameters u and v, the reciprocals of those radii will satisfy the same partial differential equations, (E_{01}) and (E_{02}), as the reciprocals of the principal radii of curvature of the original surface.

From (110) we have

$$\frac{\sqrt{E}}{\rho_1}=\frac{\sqrt{E_0}}{\rho_1^0},$$
$$\frac{\sqrt{G}}{\rho_2}=\frac{\sqrt{G_0}}{\rho_2^0}.$$

Then, the equations

$$\frac{\partial^2}{\partial u\partial v}\frac{\sqrt{E}}{\rho_1}-\frac{\partial}{\partial u}\log\frac{\sqrt{E}}{R_2}\frac{\partial}{\partial v}\frac{\sqrt{E}}{\rho_1}-\frac{\sqrt{EG}}{R_1R_2}\cdot\frac{\sqrt{E}}{\rho_1}=0,$$
$$\frac{\partial^2}{\partial u\partial v}\frac{\sqrt{G}}{\rho_2}-\frac{\partial}{\partial v}\log\frac{\sqrt{G}}{R_1}\frac{\partial}{\partial u}\frac{\sqrt{G}}{\rho_2}-\frac{\sqrt{EG}}{R_1R_2}\cdot\frac{\sqrt{G}}{\rho_2}=0,$$

do not change form when we pass to a parallel surface.

10. (a) For the ellipsoid we have $E=\frac{u(u-v)}{f(u)}$, $G=\frac{v(v-u)}{f(v)}$. Then equations (E_{01}) and (E_{02}) become

$$\left.\begin{aligned}\frac{\partial^2\varphi_{01}}{\partial u\partial v}-\frac{1}{2(u-v)}\frac{\partial\varphi_{01}}{\partial u}+\frac{3}{2(u-v)}\frac{\partial\varphi_{01}}{\partial v}=0,\\ \frac{\partial^2\varphi_{02}}{\partial u\partial v}-\frac{3}{2(u-v)}\frac{\partial\varphi_{02}}{\partial u}+\frac{1}{2(u-v)}\frac{\partial\varphi_{02}}{\partial v}=0;\end{aligned}\right\} \tag{119}$$

and the equation satisfied by (x, y, z) is

$$\frac{\partial^2\varphi_0}{\partial u\partial v}+\frac{1}{2(u-v)}\frac{\partial\varphi_0}{\partial u}-\frac{1}{2(u-v)}\frac{\partial\varphi_0}{\partial v}=0. \tag{120}$$

We can obtain a great number of particular integrals of these equations. For example, we can find solutions which are the product of a function of u by a function of v.

If a denotes a constant, we write

$$\begin{aligned}\varphi_{01}&=(u-a)^{-\frac{1}{2}}(v-a)^{-\frac{1}{2}}\\ &=\frac{1}{\sqrt{(u-a)^3(v-a)}},\\ \varphi_{02}&=(u-a)^{-\frac{1}{2}}(v-a)^{-\frac{3}{2}}\\ &=\frac{1}{\sqrt{(u-a)(v-a)^3}}.\end{aligned}$$

By using a known property of these equations we have also

$$\begin{aligned}\varphi_{01}&=(v-u)^{-1}(u-a)^{-\frac{1}{2}}(v-a)^{\frac{1}{2}}\\ &=\frac{1}{v-u}\sqrt{\frac{v-a}{u-a}},\\ \varphi_{02}&=(v-u)^{-1}(u-a)^{\frac{1}{2}}(v-a)^{-\frac{1}{2}}\\ &=\frac{1}{v-u}\sqrt{\frac{u-a}{v-a}}.\end{aligned}$$

We can find new solutions by operating on those already found with the symbols

$$\frac{\partial}{\partial u}+\frac{\partial}{\partial v},$$

$$u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v},$$

$$u^2\frac{\partial}{\partial u}+v^2\frac{\partial}{\partial v}+\tfrac{3}{2}u+\tfrac{1}{2}v.$$

We have also the general formulae

$$\varphi_{01} = \int_u^v f(x)(x-u)^{-\frac{1}{2}}(v-x)^{-\frac{1}{2}}\,dx + (v-u)^{-1}\int_u^v f_1(x)(x-u)^{-\frac{1}{2}}(v-x)^{\frac{1}{2}}\,dx,$$
$$\varphi_{02} = \int_u^v f(x)(x-u)^{-\frac{1}{2}}(v-x)^{-\frac{1}{2}}\,dx + (v-u)^{-1}\int_u^v f_1(x)(x-u)^{\frac{1}{2}}(v-x)^{-\frac{1}{2}}\,dx,$$

which contain two distinct arbitrary functions. In the same way we can obtain integrals of equation (120). The integration of this equation will determine the surfaces which admit for spherical representation of their lines of curvature a system of spherical homofocal ellipses. If u and v denote the parameters of these ellipses and λ, μ, ν the coordinates of a point of the sphere, we have

$$\lambda^2 = \frac{(a-u)(a-v)}{(a-b)(a-c)},$$
$$\mu^2 = \frac{(b-u)(b-v)}{(b-a)(b-c)},$$
$$\nu^2 = \frac{(c-u)(c-v)}{(c-a)(c-b)},$$

and the values of λ, μ, ν will satisfy equation (120). The solution

$$\varphi_0 = c + c'(u+v),$$

will give the surface of the fourth class defined by the equation

$$p = \frac{a\lambda^2 + b\mu^2 + c\nu^2}{2},$$

where λ, μ, ν are the direction cosines of the normal, and the coordinates of the point of contact of the tangent plane have the values

$$x = (p-a)\lambda, \quad y = (p-b)\mu, \quad z = (p-c)\nu.$$

If we take the solution

$$\varphi_0 = c\sqrt{(u-a)(v-a)},$$

we obtain the surfaces of the second degree.

Equation (120) also admits as particular solutions the five pentaspherical coordinates of the cyclides defined by the formula

$$x_i = \sqrt{\frac{(a_i-a)(a_i-u)(a_i-v)}{f'(a_i)}}. \qquad (i = 1, 2, 3, 4, 5).$$